# Existence and Uniqueness of Solutions of first order differential Equations

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# Existence and Uniqueness of Solutions of first order differential Equations

## Learning Outcomes

After studying this topic, we can

1. Resolve the differential equations into rational and solve it.
2. Solve equations for $p$, $x$ and $y$.
3. Explain Clairaut's equation and find its solution.
4. Find the singular solution of the differential equation.
5. Find the envelopes and orthogonal trajectories of the family of different surfaces.
6. State the existence and uniqueness theorem for first order Ordinary differential equations and use it to know whether the solution exist and unique or not.

## Introduction

We have used many methods to solve differential equations of first order and of degree 1 e.g. differential equations that can be separated in different variables, exact differential equations, equations that can be reduced to homogeneous equations and those that become exact when we multiply them by some integrating factor. In this topic we will keep our discussion on the differential equations which are of first order but of higher degree. Let us take $\frac{dy}{dx} = p$, then the general form of the first order and $n^{th}$ degree differential equation is given by the equation

$$p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_n = 0 \quad \text{—— (1)}$$

where $A_1, A_2, \ldots, A_n$ are functions of $x$ and $y$.

It is not simple to find the solution of equation (1) in its general form. In this topic we consider only that type of equation ① which we can solve easily and explain the methods for solving those equations. Also we shall discuss the Clairaut's equation, the singular solution of the differential equations, the envelopes and orthogonal trajectories of the family of surfaces and state the existence and uniqueness theorem for first order ordinary differential equations.

## Equations which can be factorized

The general form of the first order and $n^{th}$ degree differential equation is given by $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_n = 0$ where $A_1, A_2, \ldots, A_n$ are the functions of variables $x$ and $y$.

Now there are two possibilities:

(a) If we resolve $p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_n$ into rational factors of degree 1 then it can be written as $(p-f_1)(p-f_2)\ldots(p-f_n) = 0$ ② where $f_1, f_2, \ldots, f_n$ are functions of variables $x$ and $y$. Since all those values of $y$, for which the factors in equation ② become zero will satisfy equation ①. Hence to solve equation ① we will have to equate each of the factors given in equation ② to zero ie

$$p - f_r = 0 \; ; \; r = 0, 1, 2, \ldots, n \quad \text{—— ③}$$

If $F_r(x, y, c_r) = 0 \; ; \; r = 0, 1, 2, \ldots, n$ —— ④

where $c_r$; $r = 1, 2, \ldots, n$ are arbitrary constants are the solutions for eqⁿ ① is given by

$$F_1(x,y,c_1) \cdot F_2(x,y,c_2) \ldots \ldots F_n(x,y,c_n) = 0 \quad \text{—— ⑤}$$

All the constants in eqⁿ ⑤ namely $c_1, c_2, \ldots, c_n$ can have infinite number of values so all these solutions given by equation ⑤ will remain general even if we take $c_1 = c_2 = \cdots = c_n = c$

Hence the general solution is given by

$$F_1(x,y,c)\, F_2(x,y,c) \ldots \ldots F_n(x,y,c) = 0 \quad \text{—— ⑥}$$

(b) When the left hand side of equation ① cannot be factorized. We will take this possibility in next section.

Since we are dealing with the first order differential equation, the general solution should contain only one arbitrary constant. There is no loss of generality by replacing the n arbitrary constant by a single arbitrary constant.

Q Solve the differential equation $x^2p^2 + xyp - 6y^2 = 0$ where $p = \frac{dy}{dx}$

Sol$^n$ we have $x^2p^2 + xyp - 6y^2 = 0$

$$p = \frac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2}$$

$$= \frac{2y}{x} \text{ or } -\frac{3y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y}{x} \text{ or } \frac{dy}{dx} = -\frac{3y}{x} \Rightarrow \frac{dy}{y} = \frac{2}{x}dx$$

$$\text{or } \frac{dy}{y} = -\frac{3}{x}dx$$

Integrating,

$$\log y = 2\log x + \log c_1 \quad \text{or} \quad \log y = -3\log x + \log c_2$$

$$y = c_1x^2 \quad \text{or} \quad y = \frac{c_2}{x^3}$$

Hence general solution is given by

$$(y - cx^2)\left(y - \frac{c}{x^3}\right) = 0$$

$$(y - cx^2)(yx^3 - c) = 0$$

Q Solve the differential equation $(x+2y)p^3 + 3(x+y)p^2 + (2x+y)p = 0$ where $p = \frac{dy}{dx}$

Sol$^n$ $(x+2y)p^3 + 3(x+y)p^2 + (2x+y)p = 0$

$$p\{(x+2y)p^2 + 3(x+y)p + (2x+y)\} = 0$$

$$p\{(x+2y)p^2 + p\{(x+2y) + (2x+y)\} + (2x+y)\} = 0$$

$$p\{(x+2y)p(p+1) + (2x+y)(p+1)\} = 0$$

$$p(p+1)\{(x+2y)p + (2x+y)\} = 0$$

$$p = 0, \quad p+1 = 0 \quad \& \quad (x+2y)p + (2x+y) = 0$$

Solving these equations, we get

$$y = c_1, \quad \frac{dy}{dx} + 1 = 0 \Rightarrow x + y = c_2 \text{ and}$$

$$(x+2y)dy + (2x+y)dx = 0$$

$$x\,dy + y\,dx + d(x^2+y^2) = 0$$

which on integration gives

$$xy + x^2 + y^2 = c_3$$

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Hence the general solution of the given equation is given by $(y-c)(x+y-c)(xy+x^2+y^2-c)=0$

Q Solve the differential equation
$$(p-2x)(p-y)=0$$

Soln we have
$$(p-2x)(p-y)=0$$
$$\frac{dy}{dx}=2x \quad \& \quad \frac{dy}{dx}=y \text{ or } \frac{dy}{y}=dx$$
Integrating, we get
$$y=x^2+c_1 \text{ and } \log y+\log c_2=x$$
$$yc_2=e^x$$
Hence the general solution is given by
$$(y-x^2-c)(cy-e^x)=0$$

Equations which cannot be factorized

Let us write equation (1) in the form of
$$f(x,y,p)=0 \quad — \quad (7)$$
Then we cannot solve equation (7) in general form. It may be solvable for $x, y, p$ or it may be of first degree in the variables $x$ and $y$.

Let us now discuss these cases one by one

<u>Equations solvable for $x$</u>

Let the equation is given by $x=f(y,p)$ — (8)

Differentiate equation (8) w.r.t $y$ we get
$$\frac{1}{p}=\phi\left(y,p,\frac{dp}{dy}\right) \quad — \quad (9)$$
When we solve this equation we will get a relation between $p$ and $y$ which can be written in the form
$$f(y,p,c)=0 \quad — \quad (10)$$
where $c$ is an arbitrary constant

Eliminating $p$ from the equations (8) and (10) we get the required solution. If we cannot eliminate $p$ from equations (8) and (10) then we obtain $x$ and $y$ in $p$ and all these together provide the required solution

Q Solve the differential equation $y = 2px - p^3y^2$

Soln We have $y = 2px - p^3y^2$

$$\Rightarrow x = \frac{y}{2p} + \frac{p^2y^2}{2}$$

Differentiate the above equation w.r.t $y$ we get

$$\frac{1}{p} = \frac{1}{2}\left(\frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy}\right) + \frac{2p\frac{dp}{dy}y^2}{2} + \frac{2p^2}{2}y$$

$$\Rightarrow \frac{1-2p^3y}{p} = \frac{dp}{dy}\,y\,\frac{(2p^3y-1)}{p^2}$$

$$\Rightarrow \frac{dp}{dy}y = -p$$

i.e $\frac{dp}{p} + \frac{dy}{y} = 0$

Integrating, we get

$\log p + \log y = \log c$

$py = c$

Eliminating $p$ from this equation and $y = 2px - p^3y^2$ we get

$$y = \frac{2c}{y}x - \frac{c^3}{y^3}y^2$$

$$y^2 = 2cx - c^3$$

__Equations solvable for y__

Let the equation be given by $y = f(x, p)$ — (11)

Differentiating (11) w.r.t $x$, we get

$$p = \phi\left(x, p, \frac{dp}{dx}\right) \quad \text{— (12)}$$

When we solve the equation we will get a relation between $p$ and $x$ which can be written in the form

$f(x, p, c) = 0$ — (13)

where $c$ is an arbitrary constant

Now eliminating $p$ between equations (11) and (13) we get the required solution. If we cannot eliminate $p$ between equations (11) and (13) then we obtain $x$ and $y$ in $p$ and all these together provide the required solution.

Q Solve the differential equation $p^3+p-e^y=0$

Sol$^n$ we have $p^3+p-e^y=0$

$$p^3+p=e^y$$

Taking log on both sides, we get

$$y=\log p+\log(p^2+1)$$

Differentiate the above equation wrt $x$, we get

$$p=\frac{1}{p}\frac{dp}{dx}+\frac{2p}{p^2+1}\frac{dp}{dx}$$

$$dx=\left(\frac{1}{p^2}+\frac{2}{p^2+1}\right)dp$$

Integrating, we get

$$x=-\frac{1}{p}+2\tan^{-1}p+C$$

Thus $x$ and $y$ given by $x=-\frac{1}{p}+2\tan^{-1}p+C$ and $y=\log p+\log(p^2+1)$ will constitute the solution.

Q Find the solution of the differential equation $y=2px+p^2y$ for $y$

Sol$^n$ we have $y=2px+p^2y$

$$\Rightarrow y(1-p^2)=2px$$

i.e $y=\frac{2p}{1-p^2}x$

Differentiate the above equation wrt $x$, we get

$$p=\frac{2p}{1-p^2}+2x\frac{1+p^2}{(1-p^2)^2}\frac{dp}{dx}$$

$$\Rightarrow p\left(1-\frac{2}{1-p^2}\right)=2x\frac{1+p^2}{(1-p^2)^2}\frac{dp}{dx}$$

i.e $-p\frac{(1+p^2)}{1-p^2}=2x\frac{1+p^2}{(1-p^2)^2}\frac{dp}{dx}$

$$\Rightarrow p=\frac{2x}{1-p^2}\frac{dp}{dx}$$

$$\Rightarrow \frac{2}{p(1-p^2)}dp+\frac{dx}{x}=0$$

Integrating, we get

$$\int\left(\frac{2}{p}+\frac{1}{1-p}+\frac{1}{1+p}\right)dp+\log x=\log C$$

or $2\log p - \log(1-p) - \log(1+p) + \log x = \log c$

$$\Rightarrow \frac{p^2 x}{1-p^2} = c$$

Eliminating $p$ from the above equation and $y = 2px + p^2y$ we get

$$y = \frac{2\sqrt{c}\,x}{\sqrt{x+c}} + \frac{cy}{x+c}$$

$$\Rightarrow y - \frac{cy}{x+c} = \frac{2\sqrt{c}\,x}{\sqrt{x+c}}$$

$$\Rightarrow \frac{xy + cy - cy}{x+c} = \frac{2\sqrt{c}\,x}{\sqrt{x+c}}$$

ie $y = (2\sqrt{c})(\sqrt{x+c})$

$$\Rightarrow y^2 = 4cx + 4c^2$$

Clairaut's Equation

The equation $y = px + f(p)$ is known as clairaut's form.

Differentiating the equation $y = px + f(p)$ with respect to $x$ we get,

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

ie $\frac{dp}{dx}(x + f'(p)) = 0$

either $x + f'(p) = 0$ or $\frac{dp}{dx} = 0$

when $\frac{dp}{dx} = 0$, $p = c$ and hence $y = cx + f(c)$

$x + f'(p) = 0$ will give a solution which is known as singular solution, we will discuss in next section.

Equation $y = cx + f(c)$ is an equation of family of straight lines. Thus, the general solution of the clairaut's equation is an equation of a family of straight lines.

Q Solve the equation $(px - y)(px + y) = 2p$

Sol$^n$ Substituting $x^2 = X$, $y^2 = Y$ and $P = \frac{dY}{dX} = \frac{2y\frac{dy}{dx}}{2x}$

$\Rightarrow P = \frac{yp}{x}$ in the equation $(px - y)(px + y) = 2p$, we get

$$\left(\frac{x^2 P}{y} - y\right)(Px + x) = \frac{2xP}{y}$$

$\Rightarrow (px-y)(p+1) = 2p$

$y = px - \frac{2p}{p+1}$

which is clairaut's equation

Hence the solution is

$y = cx - \frac{2c}{c+1}$

$\Rightarrow y^2 = cx^2 - \frac{2c}{c+1}$

Q Solve the equation

$y = xp + \frac{1}{p}$

sol$^n$ we have $y = xp + \frac{1}{p}$

which is already clairaut's equation

with $f(p) = \frac{1}{p}$ and $f'(p) = \frac{1}{p^2}$

Hence the solution is

$y = cx + \frac{1}{c}$, $c \neq 0$ is an arbitrary constant

Also $x + f'(p) = 0$

$\Rightarrow x - \frac{1}{p^2} = 0$

Eliminating $p$ between $x - \frac{1}{p^2} = 0$ and the equation $y = xp + \frac{1}{p}$

yields $y = x\frac{1}{\sqrt{x}} + \sqrt{x} = 2\sqrt{x}$

$\Rightarrow y^2 = 4x$

which is the required solution of the equation $y = xp + \frac{1}{p}$

Q Find the singular solution of the clairaut's equation

$y = px + p^2$

sol$^n$ we have clairaut's eq$^n$ $y = px + p^2$

Differentiate wrt $x$

$p = px + xp' + 2pp'$

$\Rightarrow p'(x + 2p) = 0$

either $p' = 0 \Rightarrow p = c$

or $x + 2p = 0 \Rightarrow p = -\frac{x}{2}$ or $c = -\frac{x}{2}$

Putting $p = c$ in $y = px + p^2$ gives $y = cx + c^2$

$\Rightarrow y = -\frac{x}{2}x + \frac{x^2}{4}$

$\Rightarrow y = -\frac{x^2}{4}$

## Singular Solution

Suppose the general solution of an ordinary differential equation $f(x,y,p)=0$ is $F(x,y,C)=0$ —— (14)

A singular solution is defined as neither the general solution nor a particular solution of a differential equation.

It is proved that a curve representing the singular solution is the envelope of the curves representing the general solution of the differential equation.

So to get the singular solution, find the envelope of the curves representing the general solution of the differential equation. In the case of Clairaut's form $x+f'(p)=0$ taken along with the equation $y=xp+f(p)$ on the elimination of $p$ gives the singular solution.

**Definition: C-discriminant:**

Suppose the general solution of $f(x,y,p)=0$ is $F(x,y,C)=0$. The equation obtained by eliminating $C$ from $F(x,y,C)=0$ and $\frac{\partial F}{\partial C}=0$ is called the C-discriminant of the equation.

Remark: If the general solution is a quadratic equation in $C$ i.e $AC^2+BC+D=0$, then the C-discriminant is $B^2=4AD$.

**p-discriminant**

The eliminant of $p$ from $f(x,y,p)=0$ and $\frac{\partial f}{\partial p}=0$ is called the p-discriminant of the equation.

Remark: If the given differential equation can be expressed in the form $Lp^2+Mp+N=0$ then the p-discriminant is $M^2=4LN$.

Observation:

(a) The envelope of $F(x,y,C)=0$ is the part of C-discriminant.

(b) The envelope of $F(x,y,C)=0$ is the part of p-discriminant.

(c) The envelope of $F(x,y,C)=0$ satisfies the equation $f(x,y,p)=0$

Thus a singular solution is obtained by the following method

(i) Find the C-discriminant of the given equation by eliminating $C$ between $F(x,y,C)=0$ and $\frac{\partial F}{\partial C}=0$

(ii) Find the p discriminant of the given equation by eliminating $p$ between $f(x,y,p)=0$ and $\frac{\partial f}{\partial p}=0$

(iii) Take common factor between C-discriminant and p-discriminant.

(iv) Test whether the factors in (iii) satisfy the given equation.

(v) Only those that satisfy $f(x,y,p)=0$ will constitute the singular solution.



Q Solve the differential equation $y=-xp+x^4p^2$. Also find the singular solution

Soln We have $y=-xp+x^4p^2$

Differentiate w.r.t $x$, we get

$$p=-p-x\frac{dp}{dx}+4x^3p^2+2x^4p\frac{dp}{dx}$$

$$\Rightarrow 2p-4x^3p^2+x\frac{dp}{dx}-2x^4p\frac{dp}{dx}=0$$

i.e $2p(1-2px^3)+x\frac{dp}{dx}(1-2px^3)=0$

$$\Rightarrow \left(2p+x\frac{dp}{dx}\right)(1-2px^3)=0$$

$$\therefore 2p+x\frac{dp}{dx}=0$$

$$\Rightarrow \frac{2}{x}dx+\frac{dp}{p}=0$$

Integrating, we get

$$2\log x+\log p=\log c$$

$$\Rightarrow x^2p=c$$

Eliminating $p$ from this equation and $y=-xp+x^4p^2$, we get

$$y=-x\frac{c}{x^2}+c^2$$

$$\Rightarrow c^2x-c-xy=0$$

Differentiating w.r.t $c$, we get

$$2cx-1=0$$

Putting $c=\frac{1}{2x}$ in $c^2x-c-xy=0$, we get

$$\frac{x}{4x^2}-\frac{1}{2x}-xy=0$$

$$\Rightarrow 1+4x^2y=0$$

which is a c discriminant. We can obtain the p-discriminant by eliminating $p$ from $y=-xp+x^4p^2$ and $\frac{\partial f}{\partial p}=0=-x+2x^4p$

i.e $p=\frac{1}{2x^3}$

Hence the $p$-discriminant is $y=-\frac{x}{2x^3}+\frac{x^4}{4x^6} \Rightarrow 4x^2y+1=0$.

Since the common expression between p and c discriminant is $1+4x^2y=0$ $\therefore$ $1+4x^2y=0$ is the singular solution if it satisfy the equation $y=-\frac{1}{4x^2}$ and $p=\frac{1}{2x^3}$

Substituting the value of $y$ and $p$ satisfies the equation $y = -xp^4 + x^4p^2$. Hence $1+4x^2y=0$ is a singular solution

Q If the general solution of the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0$ is $c^2 - 4Cxy - 2Cy^2 + 4x^2y^2 = 0$. Find the singular solution.

Solⁿ The equation $x^2p^2 + yp(2x+y) + y^2 = 0$ is quadratic in $p$, its $p$ discriminant is

$$y^2(2x+y)^2 - 4x^2y^2 = 0$$

$$\Rightarrow y^3(4x+y) = 0$$

The general solution

$c^2 - 4Cxy - 2Cy^2 + 4x^2y^2 = 0$ is quadratic in $C$.

Hence the $C$ discriminant is given by

$(2y^2+4xy)^2 - 16x^2y^2 = 0$ i.e. $y^3(y+4x) = 0$

The common factors are $y=0$ and $y+4x=0$. Since these factors satisfies the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0$. Hence these are the singular solutions.

Q Solve the differential equation $y = px - p^2$. Also find the singular solution

Solⁿ We have $y = xp - p^2$ which is a clairaut's equation with

$$f(p) = -p^2 \text{ and } f'(p) = -2p$$

Hence the solution is $y = Cx - C^2$ is an arbitrary constant

Also $x + f'(p) = 0 \Rightarrow x - 2p = 0$

The elimination of $p$ from the above equation and the equation $y = xp - p^2$ yields $y = x\frac{x}{2} - \frac{x^2}{4} \Rightarrow x^2 = 4y$ which is the singular solution of the given equation.

Q Solve the differential equation $x^2(y-px) = yp^2$

Solⁿ Substituting $x^2 = X$, $y^2 = Y$ and $P = \frac{dY}{dX} = \frac{2y\frac{dy}{dx}}{2x} \Rightarrow P = \frac{yp}{x}$ in the equation $x^2\left(y - \frac{Px^2}{y}\right) = y\frac{P^2x^2}{y^2}$, we get

$$x^2\left(y - \frac{Px^2}{y}\right) = \frac{yP^2x^2}{y^2} \Rightarrow Y - PX = P^2 \Rightarrow Y = PX + P^2$$

which is a clairaut's equation

Hence solution is $Y = CX + C^2$

$$\Rightarrow y^2 = Cx^2 + C^2$$

## Envelops and orthogonal trajectories

Definition: Orthogonal Trajectories: Two families of curves such that each member of one family cuts every member of the other family at right angles are called orthogonal trajectories of one another.

From the equation $f(x,y,C)=0$ representing one parameter family of curves, we can form a differential equation of the first order $F(x,y,\frac{dy}{dx})=0$ which is the differential equation of the family of curves. Replacing $\frac{dy}{dx}$ by $-\frac{1}{\frac{dy}{dx}}$ in $F(x,y,\frac{dy}{dx})=0$ we get $F\left(x,y,-\frac{1}{\frac{dy}{dx}}\right)=0$ — (15)

This is the equation of family of orthogonal trajectory

For a family of curves by the polar equation $f(r,\theta,c)=0$ we can establish a differential equation of the family $F(r,\theta,\frac{dr}{d\theta})=0$. The differential equation of orthogonal trajectory is $F\left(r,\theta,-\frac{r^2}{\frac{dr}{d\theta}}\right)=0$ — (16)

Thus if the equation of the family of curves be given, we shall first find the differential equation by differentiation and elimination of the parameter. Then find the differential equation of the trajectory by the above process and solve the differential equation and get the cartesian or polar equation of the trajectory.

Q Find the orthogonal trajectories of a family of circles which touches the given line at a given point.

Solⁿ If we take the given line as the x-axis and the given point where the line touches the circle as the origin, we get the equation of that family of circles as

$$x^2+y^2=2ky, \text{ differentiating wrt } x, \text{ we get}$$

$$2x+2yp=2kp$$

Eliminating $2k$ from these two equations, we get

$$x^2+y^2=y\left(\frac{2x+2yp}{p}\right)$$

$$\Rightarrow (x^2-y^2)p-2xy=0$$

Put $y = vx$ in the above equation, we get 

i.e $(x^2 - v^2x^2)p + 2x^2v\left(v + x\frac{dv}{dx}\right) = 0$

$$x\frac{dv}{dx} = \frac{-1+v^2}{2v} - v \Rightarrow \frac{2v\,dv}{1+v^2} + \frac{dx}{x} = 0$$

Integrating

$\log(1+v^2) + \log x = \log C$

$\Rightarrow (1+v^2)x = C$ i.e $\left(1+\frac{y^2}{x^2}\right)x = C$

$$\Rightarrow x^2+y^2 = Cx \quad \text{——— (17)}$$

The equation (17) represents the family of circles touching the y-axis and is the required equation of the orthogonal trajectory.

Q. Find the orthogonal trajectories of the family of parabolas $y = ax^2$.

Sol$^n$ we have $y = ax^2$

Differentiating wrt $x$, we get

$p = 2ax$

Eliminating $a$ from these two equations, we get

$y = \frac{p}{2x}x^2$

$\Rightarrow px - 2y = 0$

The orthogonal trajectory of the family of parabolas has the differential equation

$$\left(-\frac{1}{p}\right)x - 2y = 0 \Rightarrow x + 2yp = 0$$
$$\Rightarrow x\,dx + 2y\,dy = 0$$

which on integration gives, $\frac{x^2}{2} + y^2 = C$

The above equation represents a family of ellipses and is the required equation of the orthogonal trajectory.

<u>Existence and Uniqueness of Solutions of first order differential equations</u>

We have used different methods to get the solutions of an ordinary differential equation. While solving the differential equation we have observed that a solution may exist or may not exist. We also observed that if a solution exists it may be unique or may not be unique. We will examine under what condition does the solution of a differential equation exist and is unique? Let us consider the first order differential equation

$$\frac{dy}{dt} = f(y,t),\ y(t_0) = y_0 \quad \text{—— (18)}$$

Now the question arise that what are the conditions for an initial value problem (18) to have atleast one solution? Also we would like to know that what are the conditions for problem (18) to have a unique solution. The answers for the above question are given by the Existence uniqueness theorem.

If the function $f(y,t)$ is continuous at every points $(y,t)$ in some rectangle $D: |y-y_0|<a,\ |t-t_0|<b$ and bounded in D,

$$|f(y,t)| \leq k \quad \forall\ (y,t) \in D$$

Then equation (18) has atleast one solution $y(t)$ in the interval $|t-t_0|<h$ where $h$ is the smaller of the two numbers $a$ and $b/k$. Also if $\frac{\partial f}{\partial t}$ is continuous for all $(y,t)$ in rectangle D and bounded therein, $\left|\frac{\partial f}{\partial t}\right| \leq M \ \forall (y,t) \in D$ then solution $y(t)$ is unique in the interval $|t-t_0|<h$

* Since $y' = f(y,t)$, then from equation (19) we have $|y'| \leq m$. In other words the slope of the solution curve $y(t)$ in D is atleast $m$ and atmost $m$. Hence we can say that a solution curve that crosses through any point $(x_0, y_0)$ must lie in the region bounded by the lines having slopes $-m$ and $m$ respectively.

Q Examine whether the solution of the IVP, $\frac{dy}{dx} = \sqrt{|y|}$ when $y(0)=0$ exist and unique or not.

Sol$^n$ We have $f(x,y) = \frac{dy}{dx} = \sqrt{|y|}$, $x_0 = 0$ and $y_0 = 0$. Let us take the rectangle D with $|x-0|<a$ and $|y-0|<b$ where $a$ and $b$ are any positive numbers. Since $f(x,y)$ is continuous and bounded in the rectangle D which contains $(0,0)$. Therefore the solution of IVP exists. For uniqueness consider the condition

$$\frac{|f(x,y_2) - f(x,y_1)|}{(y_2-y_1)} = \frac{|\sqrt{|y_2|} - \sqrt{|y_1|}|}{|y_2 - y_1|}$$

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Lipschitz condition is violated for the region that contains $y=0$. Since for $y_1=0$ and $y_2>0$ we have

$$\frac{|f(x,y_2)-f(x,y_1)|}{|y_2-y_1|} = \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}}, \; y_2>0$$

and we can made it large enough as of our choice by choosing $y_2$ sufficiently small whereas condition

$$|f(x,y_2)-f(x,y_1)| \le |y_2-y_1|$$

needs that the quotient on the left hand side of the above equation does not exceed $M$ (a fixed constant)

Therefore we can say that the solution is not unique.

o Examine $\frac{dy}{dx}=y$ when $y(0)=0$ for existence and uniqueness of solutions.

Sol$^n$ We have $f(x,y)=\frac{dy}{dx}=y$, $f(x,y)=1$, $x_0=0$ and $y_0=0$

Let us take the rectangle $D$ defined by

$$D: |x-0|<a \text{ and } |y-1|<b$$

where the numbers $a$ and $b$ are positive

In the rectangle $D$ that contains the point $(0,1)$, $f(x,y)$ is continuous and bounded therein. Therefore the solution of initial value Problem exists. Also we have $f(x,y)=1$ is continuous and bounded in $D$. Therefore we can say that the solution is unique.

o Examine $\frac{dy}{dx}=f(x,y)=\begin{cases} y(1-2x) & \text{for } x>0 \\ y(2x-1) & \text{for } x<0 \end{cases}$ with $y(1)=1$

for existence and uniqueness of solutions

Sol$^n$ The function $\frac{dy}{dx}=f(x,y)=\begin{cases} y(1-2x) & \text{for } x>0 \\ y(2x-1) & \text{for } x<0 \end{cases}$ is continuous and bounded everywhere except $x=0$.

Hence the solution exist and unique.